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Short Communication

Apparently the first closed-form solution of vibrating inhomogeneous beam with a tip mass

I. Elishakoff*, V. Johnson

Department of Mechanical Engineering, Florida Atlantic University, Boca Raton, FL 33431-0991, USA

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Abstract

Whereas there are numerous papers on the free vibrations of beams of uniform or non-uniform cross-section carrying concentrated masses, the problem does not lend itself to the closed-form solution. Here such a solution is reported, apparently for the first time. The solution originally derived for the inhomogeneous beam without a concentrated mass is generalized to include a tip mass. The semi-inverse method is utilized, to achieve this goal.

1. Introduction

Vibration of beams is a classical subject. It is elucidated in books by Sekhniashvili [1] and Gorman [2]. For example, in several handbooks there are tables and charts illustrating the natural frequencies of beams under various conditions and complicating effects. The reader may consult handbooks by Ananiev [3], Blevins [4], Pilkey [5] and Karnovsky and Lebed [6]. The subject of a beam vibration with a concentrated mass has also attracted much attention. It has been studied inter alia by Lau [7] (although the title uses the term "bar" instead of that of "beam"), Lee [8], Mabie and Rogers [9], Lau [10], Liu and Huang [11], Yang [12], Maltbaek [13] and Laura [14]. Extensive discussion was conducted on the proper boundary conditions in papers by To [15], Laura [16], Jacquot [17] and To [18]. It is instructive to read the paper by Soedel [19] on

*Corresponding author. Fax: +1 407 367 2825.
E-mail address: elishako@fau.edu (I. Elishakoff).

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philosophical implications of the above discussion. The recent studies on this topic include papers by De Rosa and Nicastro [20], Rossi et al. [21], De Rosa et al. [22,23].

All the above papers reported closed-form solutions. To the best knowledge of the writers, no closed-form solution exists for the natural frequency of a beam, be it uniform or a non-uniform, homogeneous or inhomogeneous, carrying a concentrated mass. In what follows, such a solution is derived for an inhomogeneous beam with variable modulus of elasticity, resulting in variable of flexural rigidity. The mass density is assumed to be constant.

2. Analysis

The differential equation that governs the vibration of an inhomogeneous beam reads:

$$\frac{\partial^2}{\mathrm{d}x^2}\left[D(x)\frac{\partial^2 y(x,t)}{\partial x^2}\right] + \rho(x)A(x)\frac{\partial^2 y}{\partial t^2} = 0, \tag{1}$$

where

$$D(x) = E(x)I(x) \tag{2}$$

is the flexural rigidity, $E(x)$ the Young's modulus, $I(x)$ the moment of inertia, $\rho(x)$ the mass density, $A(x)$ the cross-sectional area, $y(x,t)$ the displacement, x the axial coordinate, and t the time. We consider the case in which the cross-sectional area is a constant, $A(x) = \text{const}$, $I(x) = \text{const}$, along with the mass density $\rho(x) = \text{const}$. We introduce the non-dimensional coordinate ξ:

$$\xi = x/L, \tag{3}$$

where L is the beam's length. The beam is clamped at one end, leading to boundary conditions:

$$y(x,t) = 0, \quad \partial y/\partial x = 0 \quad \text{at } x = 0 \tag{4}$$

and at the tip carries a concentrated mass M of negligible dimensions. The boundary conditions at $x = L$ read:

$$\partial^2 y/\partial x^2 = 0, \tag{5}$$

$$\frac{\partial}{\partial x}\left[D(x)\frac{\partial^2 y}{\partial x^2}\right] = M\frac{\partial^2 y}{\partial t^2} \quad \text{at } x = L. \tag{6}$$

For the beam with $D(x) = \text{const}$, condition (6) reduces to that given by Laura [16]. We seek for harmonic vibrations in time, and set

$$y(x,t) = Y(x)\sin\omega t, \tag{7}$$

where $Y(x)$ is the mode shape, ω the natural frequency to be determined. Substituting Eq. (7) into Eq. (1) in view of Eq. (3) we get

$$\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\left[D(\xi)\frac{\mathrm{d}^2 Y}{\mathrm{d}\xi^2}\right] - \rho A\omega^2 L^4 Y = 0. \tag{8}$$

Following Ref. [24], we seek a polynomial solution for $Y(\xi)$:

$$Y(\xi) = a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3 + \xi^4, \tag{9}$$

where four constants, namely a_0, a_1, a_2, and a_3 ought to be determined from the boundary conditions. Utilization of conditions (4) leads to

$$a_0 = a_1 = 0. \tag{10}$$

Condition (5) results in

$$a_2 + 3a_3 = -6. \tag{11}$$

In order to impose condition (6) we need an expression for $D(\xi)$. We postulate it in the form of the fourth-order polynomial:

$$D(\xi) = b_0 + b_1\xi + b_2\xi^2 + b_3\xi^3 + b_4\xi^4, \tag{12}$$

where b_0, b_1, b_2, b_3 and b_4 are constants, to be determined. We use the semi-inverse method, i.e. such values of b_j so that postulates (9) and (12) satisfy the governing differential equation (8) and remaining boundary condition (6) in exact terms. The latter condition yields:

$$\begin{aligned}(b_1 + 2b_2 + 3b_3 + 4b_4)(2a_2 + 6a_3 + 12) + (b_0 + b_1 + b_2 + b_3 + b_4)(6a_3 + 24)\\ = -M\omega^2L^3(a_2 + a_3 + 1).\end{aligned} \tag{13}$$

We solve Eqs. (11) and (13) to obtain:

$$a_2 = 3[M\omega^2L^3 - 12D(1)]/2[M\omega^2L^3 - 3D(1)], \tag{14}$$

$$a_3 = -5M\omega^2L^3 - 24D(1)/2[M\omega^2L^3 - 3D(1)], \tag{15}$$

where $D(1) = b_0 + b_1 + b_2 + b_3 + b_4$. Substitution of Eqs. (9) and (12), bearing in mind Eqs. (14) and (15) into Eq. (8) leads to fourth-order polynomial equation in terms of powers of ξ:

$$\sum_{j=0}^{4} C_j\xi^j = 0, \tag{16}$$

where

$$C_0 = 24b_0 - 6b_1(5M\omega^2L^3 - 24B)/F + 6b_2(M\omega^2L^3 - 12B)/F, \tag{17}$$

$$C_1 = 72b_1 - 18b_2(5M\omega^2L^3 - 24B)/F + 18b_3(M\omega^2L^3 - 12B)/F, \tag{18}$$

$$\begin{aligned}C_2 = 144b_2 - 36b_3(5M\omega^2L^3 - 24B)/F + 36b_4(M\omega^2L^3 - 12B)/F\\ - 3\rho A\omega^2L^4(M\omega^2L^3 - 12B)/2F,\end{aligned} \tag{19}$$

$$C_3 = 240b_3 - 60b_4(5M\omega^2L^3 - 24B)/F - \rho A\omega^2L^4(5M\omega^2L^3 - 24B)/2F, \tag{20}$$

$$C_4 = 360b_4 - \rho A\omega^2L^4 = 0, \tag{21}$$

where

$$B = D(1) = b_0 + b_1 + b_2 + b_3 + b_4, \tag{22}$$

$$F = M\omega^2L^3 - 3B. \tag{23}$$

Since Eq. (16) must be valid for every value of ξ, all coefficients C_j in front of ξ^j must vanish $(j = 0, 1, 2, 3, 4)$.

The expression C_4 in front of ξ^4 is $C_4 = 360b_4 - \rho A\omega^2L^4 = 0$.

This equation gives us the formula for the natural frequency squared:

$$\omega^2 = 360b_4/\rho A L^4. \tag{24}$$

This expression coincides with the value of the natural frequency squared derived from inhomogeneous beams without concentrated mass by Elishakoff and Candan [25]. This means that when the mode shape and flexural rigidity are sought in terms of polynomial functions, the expression for the natural frequency is unaffected by the value of the concentrated mass M. The natural question arises: What is the effect of the concentrated mass if the analytical expression for the natural frequency remains unchanged by the presence of the concentrated mass? The answer is that, the expressions for the coefficients b_j of the flexural rigidity are affected by the concentrated mass.

We introduce the ratio of the concentrated mass to the mass of the entire beam:

$$\alpha = M/\rho A L. \tag{25}$$

Substitution of Eqs. (24) and (25) into the expression for C_3 results in

$$240b_3 + 120b_4(1800\alpha b_4 - 24B)/360\alpha b_4 - 3B = 0. \tag{26}$$

This equation is linear in terms of b_0, allowing expressing b_0 in terms of the rest of the coefficients b_j:

$$b_0 = b_3b_4(120\alpha - 5) - b_1b_3 - b_2b_3 - b_3^2 - 4b_4(b_1 + b_2) + b_4^2(300\alpha - 4)/b_3 + 4b_4. \tag{27}$$

Substitution of the expression into that of C_2 gives

$$2b_2b_4 + b_3^2 + 14b_3b_4 + 28b_4^2 = 0, \tag{28}$$

which is linear in terms of b_2. We solve it out for b_2 to get

$$b_2 = -(b_3^2 + 14b_3b_4 + 28b_4^2)/2b_4. \tag{29}$$

Substitution of this expression into C_1 yields:

$$4b_1b_4^2 - b_3^2 - 16b_3^2b_4 - 32b_3b_4^2 = 0, \tag{30}$$

which is linear in terms of b_1. We express b_1:

$$b_1 = b_3(b_3^2 + 16b_3b_4 + 32b_4^2)/4b_4^2. \tag{31}$$

Substitution into expression (17) for C_0 gives

$$20b_4b_3^4 + 100b_3^3b_4^2 + 240b_3^2b_4^3 + 600b_3b_4^2 + 864b_4^5 + b_3^5 + 960\alpha b_3b_4^4 + 2400\alpha b_4^5 = 0. \tag{32}$$

This is a quintic equation for either b_3 or b_4. At this stage we note that the unknowns are ω^2, b_0, b_1, b_2, b_3 and b_4. Thus, we have six unknowns; the number of equations $C_j = 0, (j = 0, 1, 2, 3, 4)$ is five.

We conclude at this stage that without setting an additional constraint we get infinite number of solutions. Let us first express these solutions in terms of a single parameter b_4, which will be treated as an arbitrary parameter. Hence it makes sense to express b_3 in terms of b_4:

$$b_3 = \beta b_4. \tag{33}$$

We notice that because Eq. (32) is quintic, for a specified α and β, one has to resort to a numerical solution. However, an analytical solution is available in an *implicit* form, expressing α from Eq. (32):

$$\alpha = -\frac{(b_3^4 + 16b_3^3 b_4 + 36b_3^2 b_4^2 + 96b_3 b_4^2 + 216b_4^4)(b_3 + 4b_4)}{480b_4^4(2b_3 + 5b_4)}. \tag{34}$$

Now, setting

$$\beta = \beta^*, \tag{35}$$

where β^* is a pre-selected value, and substituting it into Eq. (34) results in the appropriate value for the mass ratio. For different values of b_4 we obtain the exact analytical expressions via use of Eqs. (27), (31) and (29).

For $\beta^* = -5$, and $b_4 = 1$ and we get $\alpha = \frac{739}{2400}$, $b_0 = \frac{473}{8}$, $b_1 = \frac{115}{4}$, $b_2 = \frac{17}{2}$. The associated flexural rigidity of the beam is (Fig. 1)

$$D(\xi) = \tfrac{473}{8} + \tfrac{115}{4}\xi + \tfrac{17}{2}\xi^2 - 5\xi^3 + \xi^4. \tag{36}$$

For $\beta^* = -6$, $b_4 = 1$ and we have $\alpha = \frac{51}{70}$, $b_0 = 106$, $b_1 = 42, b_2 = 10$ (Fig. 2):

$$D(\xi) = 106 + 42\xi + 10\xi^2 - 6\xi^3 + \xi^4. \tag{37}$$

For $\beta^* = -7$, and $b_4 = 1$ and we obtained $\alpha = \frac{593}{480}$, $b_0 = \frac{1309}{8}$, $b_1 = \frac{217}{4}$, $b_2 = \frac{21}{2}$ (Fig. 3):

$$D(\xi) = \tfrac{1309}{8} + \tfrac{217}{4}\xi + \tfrac{21}{2}\xi^2 - 7\xi^3 + \xi^4. \tag{38}$$

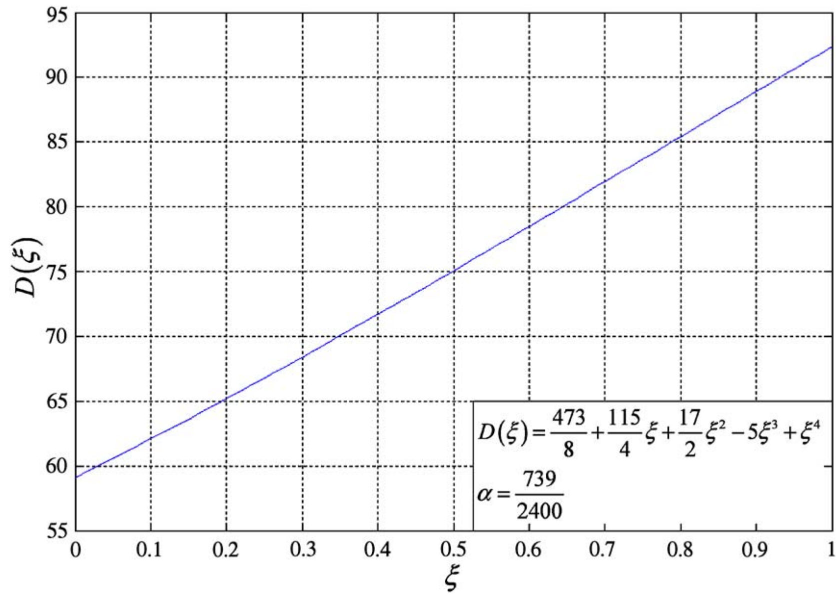

Fig. 1. Variation of flexural rigidity vs. non-dimensional axial coordinate $\beta = -5$.

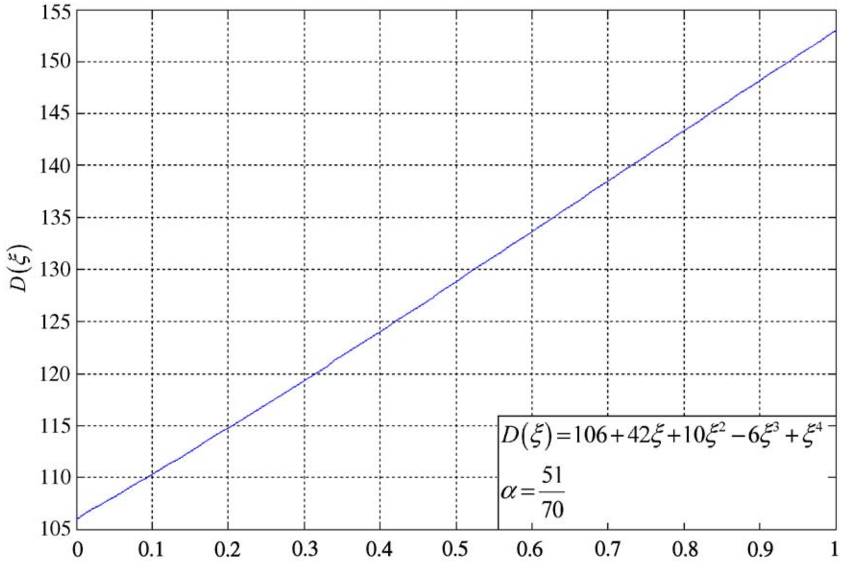

Fig. 2. Variation of flexural rigidity vs. non-dimensional axial coordinate $\beta = -6$.

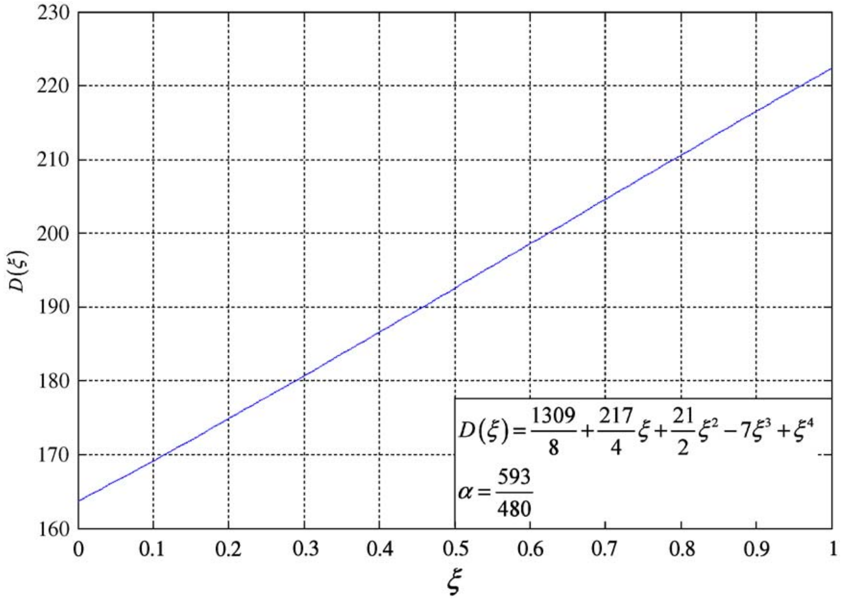

Fig. 3. Variation of flexural rigidity vs. non-dimensional axial coordinate $\beta = -7$.

For $\beta^* = -8$, and $b_4 = 1$ and we arrive at $\alpha = \frac{293}{165}$, $b_0 = 226$, $b_1 = 64, b_2 = 10$ (Fig. 4):

$$D(\xi) = 226 + 64\xi + 10\xi^2 - 8\xi^3 + \xi^4. \tag{39}$$

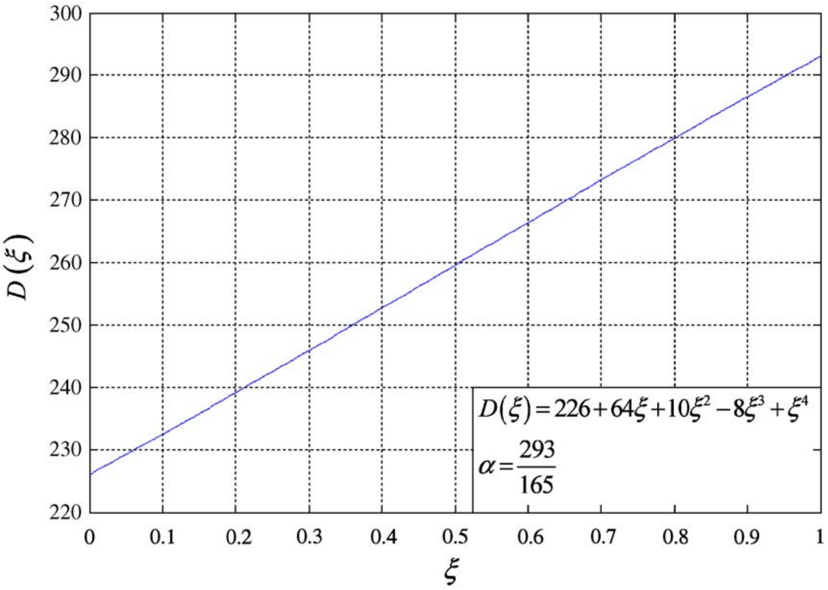

Fig. 4. Variation of flexural rigidity vs. non-dimensional axial coordinate $\beta = -8$.

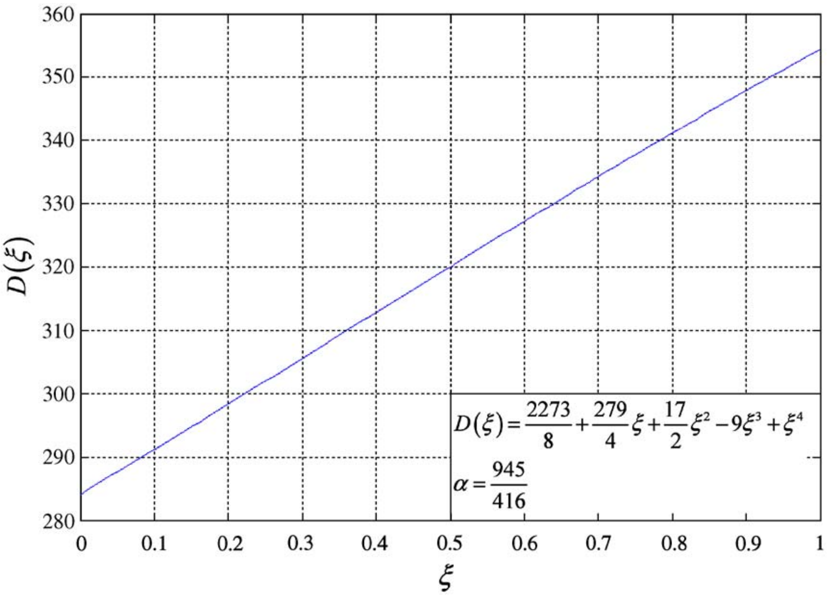

Fig. 5. Variation of flexural rigidity vs. non-dimensional axial coordinate $\beta = -9$.

For $\beta^{*} = -9$, and $b_4 = 1$ and we get $\alpha = \frac{945}{416}$, $b_0 = \frac{2273}{8}$, $b_1 = \frac{279}{4}$, $b_2 = \frac{17}{2}$ (Fig. 5):

$$D(\xi) = \frac{2273}{8} + \frac{279}{4}\xi + \frac{17}{2}\xi^2 - 9\xi^3 + \xi^4. \tag{40}$$

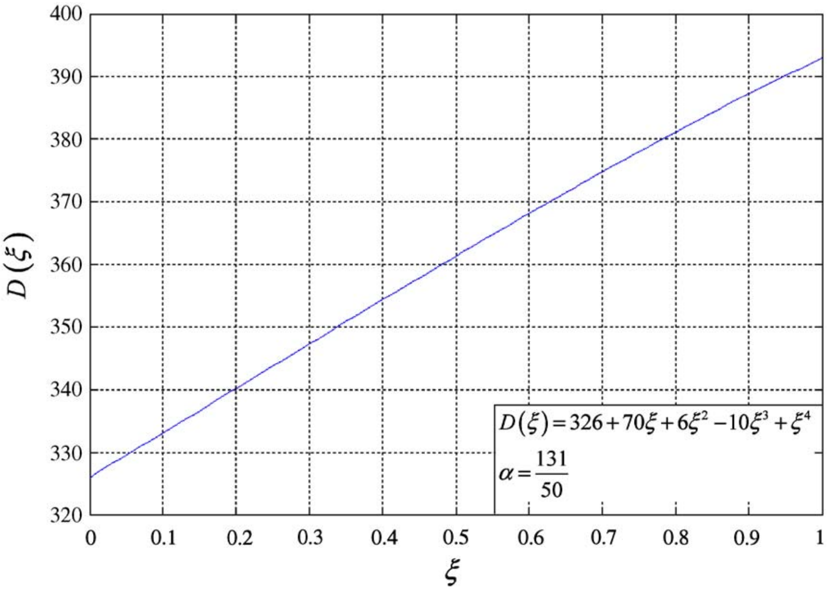

Fig. 6. Variation of flexural rigidity vs. non-dimensional axial coordinate $\beta = -10$.

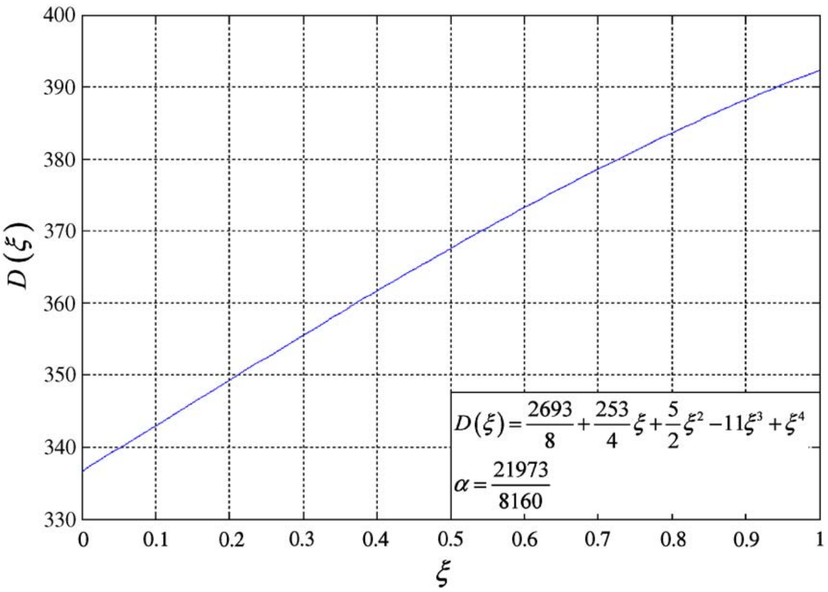

Fig. 7. Variation of flexural rigidity vs. non-dimensional axial coordinate $\beta = -11$.

For $\beta^* = -10$, and $b_4 = 1$ and we are left with $\alpha = \frac{131}{50}$, $b_0 = 326$, $b_1 = 70, b_2 = 6$ (Fig. 6):

$$D(\xi) = 326 + 70\xi + 6\xi^2 - 10\xi^3 + \xi^4. \qquad (41)$$

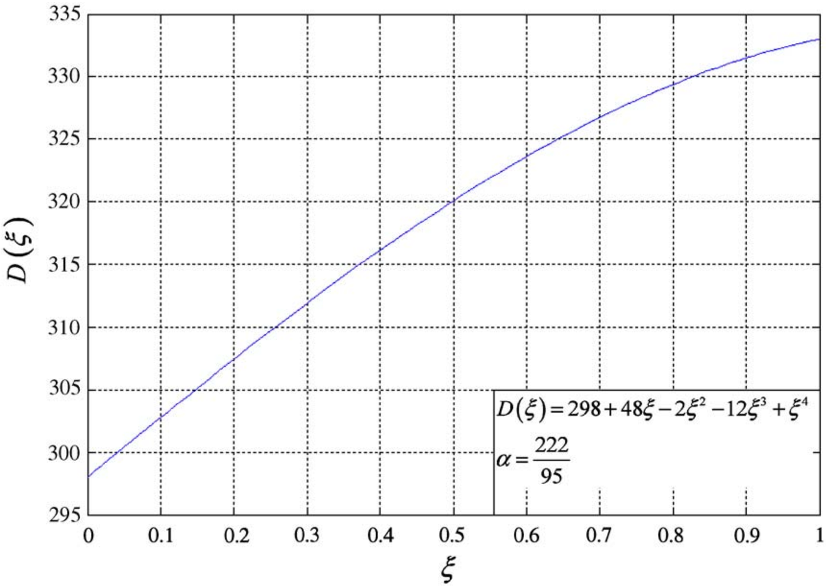

Fig. 8. Variation of flexural rigidity vs. non-dimensional axial coordinate $\beta = -12$.

For $\beta^* = -11$, and $b_4 = 1$ and we get $\alpha = \frac{21973}{8160}$, $b_0 = \frac{2693}{8}$, $b_1 = \frac{253}{4}, b_2 = \frac{5}{2}$ (Fig. 7):

$$D(\xi) = \tfrac{2693}{8} + \tfrac{253}{4}\xi + \tfrac{5}{2}\xi^2 - 11\xi^3 + \xi^4. \tag{42}$$

For $\beta^* = -12$, and $b_4 = 1$ and we obtain $\alpha = \frac{222}{95}$, $b_0 = 298$, $b_1 = 48, b_2 = -2$ (Fig. 8):

$$D(\xi) = 298 + 48\xi - 2\xi^2 - 12\xi^3 + \xi^4. \tag{43}$$

3. Discussion

In all the above cases the flexural rigidity turns out to be a positive-valued function throughout $\xi \in [0, 1]$. This is the condition needed to get a physically feasible solution. Thus, one cannot set the value of β that will result in negative flexural rigidity throughout beam's length. Unique solution can be obtained if one sets additional conditions. These can be set in different forms:

(1) One can demand that the beam with flexural rigidity in Eq. (12) should have a pre-selected solution under some static loading.

(2) One can demand that the value of the flexural rigidity at some cross-section should be equal to the pre-selected one. For example, one can demand that the flexural rigidity at the tip cross-section should equal $D(1) = B = D^*$, where D^* is a given quantity.

(3) One can demand that the natural frequency of the beam should be equal to some set value Ω. In this case, from Eq. (24) one obtains the value for b_4:

$$b_4 = \Omega^2 \rho A L^4 / 360 = b_4^*. \tag{44}$$

Once the value of b_4 is so obtained, one evaluates the values of the other coefficients in terms of the mass ratio and b_4^*.

The obtained solution can be used as a benchmark for verification of approximate or numerical techniques; additionally, when and if in the future a manufacturing method will be developed to produce an arbitrary variation of the modulus of elasticity $E(\xi)$ one will be in a position to design beams with pre-selected criterion on their performance.

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